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Vortex equation in holomorphic line bundle over complex manifold

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Abstract

In this note we study the vortex equation in holomorphic line bundle over non-Kähler complex manifolds. We prove a existence theorem to that equation by means of the upper and lower solution method to some Kazdan–Warner type equation.

Keywords: Complex manifold; Holomorphic line bundle; Vortex equation

1. Introduction

Many interesting equations in gauge theory arise as a minimizing conditions for the gauge invariant functionals. The vortex equation is one equation of this sort, and the solutions of which produce the critical points of the so-called Yang–Mills–Higgs functional. In the complex context, let M be a compact complex manifold of dimension m and E a holomorphic vector bundle on M. Then the equation can be expressed as

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$$\bar{\partial}_E \phi = 0, \qquad \sqrt{-1}\Lambda F_H + \frac{1}{2}\phi \otimes \phi^{*H} - \frac{1}{2}\tau \mathrm{I} = 0,$$

where F_H is the curvature of the Hermitian connection compatible with the $\bar{\partial}$-operator $\bar{\partial}_E$ and the Hermitian metric H on holomorphic bundle E, and $\phi \in A^0(M, E)$ is a smooth section.

The corresponding problem of finding Hermitian–Yang–Mills–Higgs metrics on holomorphic bundles with prescribed holomorphic sections over compact Kähler manifolds has been extensively discussed in [2–4,7,8] and the references therein. It was shown there that the existence of such metrics should be related to the stability property for holomorphic bundles. In this paper we are interested in the general base manifold, but with special vector bundle.

Now let M be a compact complex manifold with a Hermitian metric g. We say that the metric g is standard/Gauduchon if the associated Kähler form ω_g satisfies the condition: $\partial\bar{\partial}(\omega_g^{m-1}) = 0$. Let L be a holomorphic line bundle over M, with the holomorphic structure $\bar{\partial}_L$. Let $\phi \in A^0(M, L)$ be a holomorphic section of L and H be a given background Hermitian metric on L. In the special case of holomorphic line bundle the vortex equation can be translated into the form of the following equation:

$$\sqrt{-1}\Lambda_g \bar{\partial}\partial u + \frac{1}{2}|\phi|_H^2 e^u + \left(\sqrt{-1}\Lambda_g F_H - \frac{1}{2}\tau\right) = 0, \tag{1}$$

where $K = He^u$ is any other metric and $\tau \in \mathbb{R}$ is a real parameter.

In this note we prove the following

Theorem 1.1. *Let M be an m-dimensional compact complex manifold with standard metric g. Let L be a holomorphic line bundle over M and $\phi \in A^0(M, L)$ be a prescribed holomorphic section of L. Then there exists a Hermitian metric K on L satisfying the above vortex equation* (1) *if and only if*

$$\frac{\tau \operatorname{Vol}(M)}{4\pi} > \mu(L),$$

where $\mu(L) = c_1(L, \omega_g)$ is the slope of L with respect to the Kähler form ω_g on M.

Remark 1. The condition on the metric above is not so restrictive, since by a theorem of Gauduchon there is a standard metric in the conformal class of every Hermitian metric. The above theorem should be helpful to provide a description of the moduli space of all vortices.

2. Proof of Theorem 1.1

Let M be a complex manifold of dimension m. If M has a Hermitian metric g and ω_g is the associated Kähler form, then we can define a contraction map $\Lambda_g : A^{1,1}(M) \to A^0(M)$, $a \to \Lambda_g a$, where $A^{1,1}(M)$, respectively $A^0(M)$, is the space of $(1,1)$-forms, respectively smooth, functions, on M. We will first study the equation

$$-\sqrt{-1}\Lambda_g \bar{\partial}\partial u + he^u - c = 0, \tag{2}$$

where c is some constant.

The following proposition will imply our Theorem 1.1.

Proposition 2.1. *Let M be a compact complex manifold with standard metric g. Then the following holds*:

(i) *for $c = 0$, a necessary condition for the existence of a smooth function u to Eq.* (2) *is that h changes sign on M*;
(ii) *for $c > 0$, a necessary condition for the existence of a function u to Eq.* (2) *is that h is strictly positive somewhere on M*;
(iii) *for $c < 0$, if $h \leqslant 0$, then there is a $u \in C^\infty(M)$ which satisfies Eq.* (2).

Remark 2. When (M, g) is a Kähler manifold, then Eq. (2) turns out to be precisely of the form considered in [1]:

$$-\Delta u + he^u - c = 0,$$

which is known as Kazdan–Warner equation. (Note that the Laplacian used by Kazdan and Warner is the negative definite operator which differs from the above by a minus sign.)

In the context of the Kazdan and Warner's Problem, i.e., to solve the last equation on a Riemannian surface, the variational method is quite powerful, since in real 2-dimensional Riemannian manifold, we have the following Moser–Trudinger inequality: $\exists C > 0$, so that $\forall u \in H_1^2(M)$ we have

$$\log \int_M e^u \, dV \leqslant \frac{1}{16\pi} \int_M |\nabla u|^2 + \int_M u \, dV + C.$$

Such a method does not work well for higher dimensional case, so here instead we shall use the method of upper and lower solutions.

As is well known [6], to solve the above Eq. (2), it is sufficient to find functions u_+, u_-, with $u_- \leqslant u_+$, such that

$$-\sqrt{-1}\Lambda_g \bar{\partial}\partial u_- + he^{u_-} - c \geqslant 0 \quad \text{and}$$
$$-\sqrt{-1}\Lambda_g \bar{\partial}\partial u_+ + he^{u_+} - c \leqslant 0,$$

since these imply the existence of a solution u, $u_- \leqslant u \leqslant u_+$.

The following lemma holds:

Lemma 2.2 [5]. *Let M be a compact complex manifold of dimension m with Hermitian metric g. If g is standard/Gauduchon, then for given $f \in C^\infty(M)$, the equation $-\sqrt{-1}\Lambda_g \bar{\partial}\partial(u) = f$ has a solution $u \in C^\infty(M)$ if and only if $\int_M f \, \omega_g^m = 0$.*

The following lemma shows that it is easy to find a lower solution.

Lemma 2.3. *Given any $h \in L^p(M)$ (where $p \geqslant 2m + 1$) and a function $u_+ \in H^{2,p}(M)$, there is a lower solution $u_- \in H^{2,p}(M)$ of* (2) *with $c < 0$ such that $u_- \leqslant u_+$.*

Proof. Let $k_1(x) = \max(1, -h(x))$ and let $\alpha > 0$ be a constant chosen so that $\alpha\bar{k_1} = -c$, where $\bar{k_1}$ denotes the mean value of k_1. Then $\overline{(\alpha k_1 + c)} = 0$ and $(\alpha k_1 + c) \in L^p(M)$. Thus there is a solution v of $-\sqrt{-1}\Lambda_g \partial\bar{\partial} v = (\alpha k_1 + c)$, by Lemma 2.2 above. By the L^p-regularity theory, $v \in H^{2,p}(M)$ and hence v is continuous. We claim that by choosing the constant β sufficiently large, the function $u_- = v - \beta$ meets our requirements. One can clearly satisfy $u_- \leqslant u_+$ for any $u_+ \in H^{2,p}(M)$, because v and u_+ are continuous. In addition, u_- is a lower solution since

$$-\sqrt{-1}\Lambda_g \bar{\partial}\partial u_- - c + he^{u_-} = \alpha k_1 + he^{v-\beta} \geqslant k_1(\alpha - e^{v-\beta}) > 0$$

for β sufficiently large. □

Lemma 2.4. *With the same assumption as in above and if $h(x) \leqslant 0$ for all $x \in M$ but $h \neq 0$, then Eq.* (2) *has a solution.*

Proof. In view of the above Lemma 2.3, it suffices to find an upper solution u_+ of the equation. Let $-\sqrt{-1}\Lambda_g \bar{\partial}\partial v = \bar{h} - h$, and note that $\bar{h} < 0$. Pick constants a and b so large that $a\bar{h} < c$ and $(e^{av+b} - a) > 0$. Then let $u_+ = av + b$. Since $h \leqslant 0$, we have

$$\begin{aligned} -\sqrt{-1}\Lambda_g \bar{\partial}\partial u_+ - c + he^{u_+} &= a\bar{h} - ah - c + he^{av+b} \\ &= (a\bar{h} - c) + h(e^{av+b} - a) < 0. \end{aligned}$$

Therefore u_+ is an upper solution. This completes the proof of this lemma. □

Proof of Proposition 2.1. Parts (i) and (ii) are clear from the metric condition and the theorem of Stokes. (iii) was already obtained in the above lemmas. □

Proof of Theorem 1.1. Fix a background metric H on L. Let $K = He^u$ be another metric and let

$$c = \int_M \left(\sqrt{-1}\Lambda F_H - \frac{1}{2}\tau\right),$$

and choose $v \in C^\infty(M, \mathbb{R})$ to be the solution to

$$-\sqrt{-1}\Lambda_g \bar{\partial}\partial v = \left(\sqrt{-1}\Lambda F_H - \frac{1}{2}\tau\right) - c.$$

Define

$$w = u - v.$$

Then u is a solution to (1) if and only if w is a solution to

$$-\sqrt{-1}\Lambda_g \bar{\partial}\partial w - \left(\frac{1}{2}|\phi|_H^2 e^v\right)e^w - c = 0.$$

The vortex equation then becomes

$$-\sqrt{-1}\Lambda_g \bar{\partial}\partial w + he^w - c = 0, \tag{3}$$

with

$$h = -\left(\frac{1}{2}|\phi|_H^2 e^v\right).$$

By Proposition 2.1, Eq. (3) has a (unique) solution if and only if $c < 0$, that is,

$$\int_M \left(\sqrt{-1}\Lambda F_H - \frac{1}{2}\tau\right) < 0.$$

The result now follows from the Chern–Weil formula for $c_1(L, \omega_g)$. $\square$

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